Math 142 Lecture 1 Notes

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1 Introduction to Topology

1.1 Motivation and overview

What is topology? It is not geometry. Geometry is the study of "rigid" objects, distances, curvature, and symmetries/isometries. On the other hand, topology is the study of "non-rigid" phenomena, connectivity, and deformation (stretching, squeezing, etc.).

Example 1.1. Consider the difference between the surface of a tennis ball and the surface of a soccer ball. Geometrically, the surfaces have different properties, but a topological point of view would call them similar, or "the same."

Example 1.2. Consider the difference between A, $\mathcal{A}$, and $\mathscr{A}$. Geometrically, these are different shapes, but you might think of them all as the letter A. There is some common property they all share that makes them appear like the shape of the letter A.

Example 1.3. Are the letters V and X the the same, topologically? Maybe not. You can remove a point from the X and get 4 pieces, but you cannot do that with the V, no matter how you stretch it.

What does "the same" mean? We will see two approaches to this:

1. "homeomorphism" (think reparameterising)
2. "homotopy equivalent" (think same number of holes).

Example 1.4. A metal washer and a toilet paper roll might be considered to have the same number of holes.

Where does algebra come in? The idea is to encode information about your space using algebra.

Example 1.5. We will find a map G that takes a topological space and associates a group. Ideally, we want this to "respect" maps between spaces. So if $f : X \to Y$ is a continuous map, then $f_* : G(X) \to G(Y)$ will be a homomorphism. We also want composition to carry through; i.e. if $f : X \to Y$ and $g : Y \to Z$ are continuous, then $(g \circ f)_* = g_* \circ f_*$.

In this course, we will see two algebraic invariants:

1. Fundamental group $\pi_1(X)$
2. Homology groups $H_*(X)$.

They will fit in to the following outline of the course (probably):

1. Point-set topology (up to midterm 1)
2. Fundamental group
3. Homology
4. Applications.

There are other approaches to topology. For example, people study differential topology and analysis on topological spaces, which are both rich fields in their own right.

1.2 Topological spaces

Definition 1.1. Let X be a set. Then a *topology* on X is a collection of subsets of X, called *open sets*, such that

1. The sets $\varnothing$ and X are both open.
2. Any union of open sets is an open set (even of uncountably infinitely many).
3. The intersection of finitely many open sets is open.

The set X, along with its topology is called a *topological space*.

Example 1.6. This is the usual topology on $\mathbb{R}^n$ (also called $\mathbb{E}^n$).[1] Call a set $A \subseteq \mathbb{R}^n$ open iff for all $x \in A$, we can find an $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq A$, where $B_\varepsilon(x)$ is the ball of radius ε centered at x.

Let's check the properties.

1. $\varnothing$ is vacuously open, and $\mathbb{R}^n$ itself is open because for $x \in \mathbb{R}^n$, $B_\varepsilon(x) \subseteq \mathbb{R}^n$ for all ε.
2. If $A = \bigcup A_i$, then if $x \in A_i$ for some i. A_i is open so there exists some $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq A_i$. So $B_\varepsilon(x) \subseteq A_i \subseteq \bigcup A_i = A$.
3. If $A = \bigcup_{i=1}^n A_i$ and $x \in A$, then $a \in A_i$ for $i = 1, \ldots, n$. So for each i, there exists ε_i such that $B_{\varepsilon_i}(x) \subseteq A_i$. Let $\varepsilon = \min(\varepsilon_1, \ldots, \varepsilon_n)$. Then $B_\varepsilon(x) \subseteq B_{\varepsilon_i} * x(\subseteq A_i$ for all i. So $B_\varepsilon(x) \subseteq \cap A_i = A$ making A open.

[1] The letter E here is for Euclidean.

Example 1.7. A metric space (X, d) automatically has a topology *induced* by the metric. For $x \in X$, define $B_\varepsilon(x) = \{y \in X : d(x,y) \le \varepsilon\}$.[2] Then define the topology on X as in the $\mathbb{R}^n$ example.

Remark 1.1. Different metrics might give the same topology.

Topologies induced by metrics are easier to visualize. However, there are "weirder" topologies that do not necessarily correspond to a metric.

Example 1.8. Let X be an space, and let the open sets be $\{\varnothing, X\}$. This is called the *trivial* or *indiscrete topology.*

Example 1.9. Let X be any space, say that every subset of X is open. This is called the *discrete topology.*

Example 1.10. If X is a topological space and $Y \subseteq X$, then the *subspace* (or *induced*) *topology* on Y has $A \subseteq Y$ open iff $A = Y \cap U$ for some $U \subseteq X$ open.

Definition 1.2. Let X be a topological space. A set $B \subseteq X$ is *closed* if $X \setminus B$ is open.[3]

Example 1.11. Both $\varnothing$ and X are closed, in addition to being open.

Definition 1.3. If $x \in X$, a *neighborhood* of x is any open set $A \subseteq X$ with $x \in A$.[4]

How do we show a set A is closed? Show that $X \setminus A$ is open. How do we show that a set A is open? For every $x \in A$, find a neighborhood U_x such that $U_x \subseteq A$. Then $A = \bigcup_{x \in A} U_x$ is a union of open sets, so it is open.

[2] We could have used $<$ here instead of $\le$, but it does not matter because they produce the same topology.
[3] This is sometimes called $X - B$, and is $\{x \in X : x \notin B\}$.
[4] This term adds nothing new, but it is shorter and cleaner to say and write. Otherwise, we would always have to talk about "an open set containing x."